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Fermionic construction of partition function for multi-matrix models and multi-component TL hierarchy[1]

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Abstract

We use p-component fermions ($p = 2, 3, \dots$) to present $(2p-2)N$-fold integrals as a fermionic expectation value. This yields fermionic representation for various $(2p-2)$-matrix models. Links with the p-component KP hierarchy and also with the p-component TL hierarchy are discussed. We show that the set of all (but two) flows of p-component TL changes standard matrix models to new ones.

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1 Introduction

Let $d\mu_\alpha(x,y)$ be a set of measures (in general, complex), supported on a finite set of products of curves in the complex x and y planes.

Let ρ_α, $\alpha = 2,\ldots,p-1$, be a set of functions in two variables.

Let $x^{(\alpha)} = (x_1^{(\alpha)},\ldots,x_N^{(\alpha)})$ and $y^{(\alpha)} = (y_1^{(\alpha)},\ldots,y_N^{(\alpha)})$, $\alpha = 1,\ldots,p$ are two sets of variables, where $x^{(p)}$ and $y^{(1)}$ are fixed by

$$x_i^{(p)} = y_i^{(1)} = N-i, \quad i = 1,\ldots,N \tag{1.1}$$

We shall use the following notation

$$d\mu_\alpha(x^{(\alpha)}, y^{(\alpha+1)}) := \prod_{i=1}^{N} d\mu_\alpha(x_i^{(\alpha)}, y_i^{(\alpha+1)}) \tag{1.2}$$

We consider the following integral over $(2p-2)N$ variables $x^{(\alpha)} = (x_1^{(\alpha)},\ldots,x_N^{(\alpha)})$ and $y^{(\alpha+1)} = (y_1^{(\alpha+1)},\ldots,y_N^{(\alpha+1)})$, $\alpha = 1,\ldots,p-1$:

$$Z_N = \int \prod_{\alpha=1}^{p} \varrho_\alpha(y^{(\alpha)}, x^{(\alpha)}) \prod_{\alpha=1}^{p-1} d\mu_\alpha(x^{(\alpha)}, y^{(\alpha+1)}) \tag{1.3}$$

where

$$\varrho_1(y^{(1)}, x^{(1)}) = \det\left(\left(x_i^{(1)}\right)^{y_j^{(1)}}\right)_{i,j=1,\ldots,N} = \prod_{i>j}^{N}(x_i^{(1)} - x_j^{(1)}) =: \Delta_N(x^{(1)}),$$

$$\varrho_p(y^{(p)}, x^{(p)}) = \det\left(\left(y_i^{(p)}\right)^{x_j^{(p)}}\right)_{i,j=1,\ldots,N} = \prod_{i>j}^{N}(y_i^{(p)} - y_j^{(p)}) =: \Delta_N(y^{(p)})$$

are Vandermonde determinants, and where

$$\varrho_\alpha(y^{(\alpha)}, x^{(\alpha)}) = \det\left(\rho_\alpha(y_i^{(\alpha)}, x_j^{(\alpha)})\right)_{i,j=1,\ldots,N}, \quad \alpha = 2,\ldots,p-1 \tag{1.4}$$

Developing each $\det\rho_\alpha$ into $N!$ monomial terms (each is labeled by an element of the permutation group S_N), and, for given choice of the element of the permutation group, say σ, using the change of variables inside of each N-fold integral to the left (namely, $x^{(\beta)} \to \sigma(x^{(\beta)})$, $y^{(\beta)} \to \sigma(y^{(\beta)})$ for all $\beta \leq \alpha$), then, using the anti-symmetry of $\Delta_N(x^{(1)})$

(which is the integrand of the very left N-fold integral), one finds that each term of the mentioned development yields the same contribution. This is a standard way to re-write (1.3) as

$$Z_N = \frac{1}{(N!)^{2p-2}} \int \dots \int \Delta_N(x^{(1)}) \prod_{i=1}^{N} d\mu_1(x_i^{(1)}, y_i^{(2)}) \rho_2(y_i^{(2)}, x_i^{(2)})$$

$$\int \dots \int \prod_{i=1}^{N} \rho_3(y_i^{(3)}, x_i^{(3)}) d\mu_2(x_i^{(2)}, y_i^{(3)})$$

$$\dots$$

$$\int \dots \int \prod_{i=1}^{N} \rho_{p-1}(y_i^{(p-1)}, x_i^{(p-1)}) d\mu_{p-2}(x_i^{(p-2)}, y_i^{(p-1)})$$

$$\int \dots \int \prod_{i=1}^{N} d\mu_{p-1}(x_i^{(p-1)}, y_i^{(p)}) \Delta_N(y^{(p)})$$

Integrals (1.3) may be related to the so-called determinantal ensembles [6].

For special choice of measures $d\mu_\alpha$ and functions ρ_α, integrals (1.3) arose in the study of multi-matrix models, where matrices $M_1, M_2, M_3, \dots, M_{2p-2}$ with eigenvalues respectively equal to the sets $\{x_i^{(1)},\ i = 1, \dots, N\}$,$\{y_i^{(2)},\ i = 1, \dots, N\}$,$\{x_i^{(2)},\ i = 1, \dots, N\}$,...,$\{y_i^{(p)},\ i = 1, \dots, N\}$, are coupled in an open chain. It occurs in case when one can reduce the integration over matrix entries to the integrals over eigenvalues of each matrix (for these topic see [16], [17] and Appendices to [18], [1]). Depending on $d\mu_\alpha$ and functions ρ_α, these are models of normal matrices, and certain models of random Hermitian (anti-Hermitian) matrices and certain models of random unitary matrices, see [4], [5], [11], [3], [8], [16], [17], together with discrete versions of these matrix models [18].

For instance, to obtain the partition function for the model of random N by N Hermitian matrices, $M_1, \dots, M_{2p-2}$, coupled in a chain,

$$\int e^{\mathrm{T}r \sum_{k=1}^{2p-2} V_k(M_k) + \mathrm{T}r(c_1 M_1 M_2 + \dots + c_{2p-3} M_{2p-3} M_{2p-2})} \prod_{k=1}^{2p-2} dM_k,$$

one takes

$$d\mu_\alpha(x, y) = e^{c_{2\alpha-1}xy + V_{2\alpha-1}(x) + V_{2\alpha}(y)}, \quad \alpha = 1, \dots, p-1 \tag{1.5}$$

$$\rho_\alpha(x, y) = e^{c_{2\alpha}xy}, \quad \alpha = 2, \dots, p-1 \tag{1.6}$$

Then $x_i^{(\alpha)}$, $i = 1, \dots, N$, are eigenvalues of Hermitian matrices with odd numbers, say $M_{2\alpha-1}$, while $y_i^{(\alpha)}$, $i = 1, \dots, N$, are eigenvalues of $M_{2\alpha}$, $\alpha = 1, \dots, p-1$. For future purpose, let us use the obvious freedom to re-write $d\mu_\alpha$ and ρ_α in form

$$d\mu_1(x,y) \to e^{c_1 xy + V_1(x)}, \quad d\mu_{p-1}(x,y) \to e^{c_{2p-3}xy + V_{2p-2}(x)}, \tag{1.7}$$

$$d\mu_\alpha(x,y) \to e^{c_{2\alpha-1}xy}, \ \alpha = 2, \dots, p-2, \qquad \rho_\alpha(x,y) \to e^{c_{2\alpha}xy + V(x) + V(y)}, \quad \alpha = 2, \dots, p-1 \tag{1.8}$$

In the present paper we have two tasks.

First, we equate the integral (1.3) to the fermionic vacuum expectation value. Here we use the so-called p-component fermions. This may be considered as a continuation of of the work [11].

Second, as a continuation of [7], we relate Z_N to the coupled p-component KP hierarchies, or, the same to the p component TL hierarchy. For this purpose we consider the following deformation of the first and the last measures

$$d\mu_1(x,y) \to d\mu_1(x,y|\mathbf{t}^{(1)}, n, \bar{\mathbf{t}}^{(1)}) := x^{n_1} e^{V(x,\mathbf{t}^{(\alpha)}) + V(x^{-1}, \bar{\mathbf{t}}^{(1)})} d\mu_1(x,y), \tag{1.9}$$

$$d\mu_{p-1}(x,y) \to d\mu_{p-1}(x,y|\mathbf{t}^{(p)}, n, \bar{\mathbf{t}}^{(p)}) := y^{n_p} e^{V(y,\mathbf{t}^{(p)}) + V(y^{-1}, \bar{\mathbf{t}}^{(p)})} d\mu_{p-1}(x,y) \tag{1.10}$$

$$V(x, \mathbf{t}^{(\alpha)}) = \sum_{m=1}^{\infty} x^m t_m^{(\alpha)}, \quad V(x^{-1}, \bar{\mathbf{t}}^{(\alpha)}) = \sum_{m=1}^{\infty} x^{-m} \bar{t}_m^{(\alpha)}, \quad \alpha = 1, p, \tag{1.11}$$

and also the following deformations of functions ρ_α, $\alpha = 2, \dots, p-1$,

$$\rho_\alpha(x,y) \to \tau_{n^{(\alpha)}}(\mathbf{t}^{(\alpha)} + [x], \bar{\mathbf{t}}^{(\alpha)} + [y]), \quad \alpha = 2, \dots, p-1, \tag{1.12}$$

where in the right hand side we have tau functions (labeled by $\alpha = 2, \dots, p-1$) of the one-component TL hierarchy, and where $+[x]$ and $+[y]$ denote the so-called Miwa shift of a TL (a one-component TL) higher times, details are written down below.

The deformation (1.9)-(1.12) relates integrals (1.3) to the coupled p-component KP hierarchies. If in (1.3) we take the deformed measures and the deformed functions ρ_α as described above, then, Z_N turns out to be a certain tau function of coupled p-component KP, or the same, p-component TL hierarchy, where the sets of complex numbers $\mathbf{t}^{(\alpha)} = (t_1^{(\alpha)}, t_2^{(\alpha)}, \dots)$, $\bar{\mathbf{t}}^{(\alpha)} = (\bar{t}_1^{(\alpha)}, \bar{t}_2^{(\alpha)}, \dots)$, and the set of integers $n^{(\alpha)}$, $\alpha = 1, \dots, p$, play the role of higher p-component TL times. For the sake of brevity we shall also use the notations $\mathbf{t} = (\mathbf{t}^{(1)}, \dots, \mathbf{t}^{(p)})$ and $\bar{\mathbf{t}} = (\bar{\mathbf{t}}^{(1)}, \dots, \bar{\mathbf{t}}^{(p)})$.

Important to mark, that the deformation (1.9), (1.10) and (1.12) seems do not keep the form (1.7)-(1.8). In our case the interaction $e^{c_{2\alpha} M_{2\alpha} M_{2\alpha+1} + V_{2\alpha}(M_{2\alpha}) + V_{2\alpha+1}(M_{2\alpha+1})}$ is replaced by arbitrary chosen one-component TL tau function (1.12) where $x = x^{(\alpha)}$ is the collection of eigenvalues of the matrix $M_{2\alpha}$ while $y = y^{(\alpha)}$ is the collection of eigenvalues of the matrix $M_{2\alpha+1}$.

Let us note that one can consider $(p-1)N$-fold integrals if he specifies the measures $d\mu_\alpha(x, y)$ to be proportional to Dirac delta function which equate x to a function of y (it may be $\delta(x - y)$).

The present paper is a part of series of papers devoted to fermionic approaches to multi-fold integrals, see [12], [1], [2]. Let us mark that our fermionic constructions of papers [1], [2] and of the present paper are different from what was considered in [11] and also different of [12].

1.1 Free fermions

Let $\mathcal{A}$ be the complex Clifford algebra over $\mathbb{C}$ generated by *charged free fermions* $\{f_i,\ \bar{f}_i\}_{i\in\mathbf{Z}}$, satisfying the anticommutation relations

$$[f_i, f_j]_+ = [\bar{f}_i, \bar{f}_j]_+ = 0, \quad [f_i, \bar{f}_j]_+ = \delta_{ij}. \tag{1.13}$$

Any element of the linear part

$$W := (\oplus_{m\in\mathbf{Z}} \mathbb{C} f_m) \oplus \left(\oplus_{m\in\mathbf{Z}} \mathbb{C} \bar{f}_m\right) \tag{1.14}$$

will be referred to as a *free fermion*. We also introduce the fermionic free fields

$$f(x) := \sum_{k\in\mathbf{Z}} f_k x^k, \quad \bar{f}(y) := \sum_{k\in\mathbf{Z}} \bar{f}_k y^{-k-1}, \tag{1.15}$$

which may be viewed as generating functions for the $f_j, \bar{f}_j$'s.

This Clifford algebra has a standard Fock space representation defined as follows. Define the complementary, totally null (with respect to the underlying quadratic form) and mutually dual subspaces

$$W_{an} := (\oplus_{m<0} \mathbb{C} f_m) \oplus \left(\oplus_{m\geq 0} \mathbb{C} \bar{f}_m\right), \qquad W_{cr} := (\oplus_{m\geq 0} \mathbb{C} f_m) \oplus \left(\oplus_{m<0} \mathbb{C} \bar{f}_m\right), \tag{1.16}$$

and consider the left and right $\mathcal{A}$-modules

$$F := \mathcal{A}/\mathcal{A}W_{an}, \qquad \bar{F} := W_{cr}\mathcal{A}\backslash\mathcal{A}. \tag{1.17}$$

These are cyclic $\mathcal{A}$-modules generated by the vectors

$$|0\rangle = 1 \mod \mathcal{A}W_{an}, \qquad \langle 0| = 1 \mod W_{cr}\mathcal{A}, \tag{1.18}$$

respectively, with the properties

$$\begin{array}{llll} f_m|0\rangle = 0 & (m < 0), & \bar{f}_m|0\rangle = 0 & (m \geq 0), \\ \langle 0|f_m = 0 & (m \geq 0), & \langle 0|\bar{f}_m = 0 & (m < 0). \end{array} \tag{1.19}$$

The *Fock spaces* F and $\bar{F}$ are mutually dual, with the hermitian pairing defined via the linear form $\langle 0||0\rangle$ on $\mathcal{A}$ called the *vacuum expectation value.* This is determined by

$$\langle 0|1|0\rangle = 1; \quad \langle 0|f_m\bar{f}_m|0\rangle = 1, \quad m < 0; \quad \langle 0|\bar{f}_m f_m|0\rangle = 1, \quad m \geq 0, \tag{1.20}$$

$$\langle 0|f_n|0\rangle = \langle 0|\bar{f}_n|0\rangle = \langle 0|f_m f_n|0\rangle = \langle 0|\bar{f}_m\bar{f}_n|0\rangle = 0; \quad \langle 0|f_m\bar{f}_n|0\rangle = 0, \quad m \neq n, \tag{1.21}$$

together with the Wick theorem which implies, for any finite set of elements $\{w_k \in W\}$,

$$\begin{aligned} \langle 0|w_1 \cdots w_{2n+1}|0\rangle &= 0, \\ \langle 0|w_1 \cdots w_{2n}|0\rangle &= \sum_{\sigma \in S_{2n}} sgn\sigma \langle 0|w_{\sigma(1)}w_{\sigma(2)}|0\rangle \cdots \langle 0|w_{\sigma(2n-1)}w_{\sigma(2n)}|0\rangle. \end{aligned} \tag{1.22}$$

Here σ runs over permutations for which $\sigma(1) < \sigma(2), \ldots, \sigma(2n-1) < \sigma(2n)$ and $\sigma(1) < \sigma(3) < \cdots < \sigma(2n-1)$.

Now let $\{w_i\}_{i=1,\ldots,N}$, be linear combinations of the f_j's only, $j \in \mathbf{Z}$, and $\{\bar{w}_i\}_{i=1,\ldots,N}$ linear combinations of the $\bar{f}_j$'s, $j \in \mathbf{Z}$. Then(1.22) implies

$$\langle 0|w_1 \cdots w_N\bar{w}_N \cdots \bar{w}_1|0\rangle = \det\left(\langle 0|w_i\bar{w}_j|0\rangle\right)|_{i,j=1,\ldots,N} \tag{1.23}$$

Following refs. [9], [10], for all $N \in \mathbf{Z}$, we also introduce the states

$$\langle N| := \langle 0|C_N \tag{1.24}$$

where

$$C_N := \bar{f}_0 \cdots \bar{f}^{(\alpha)}_{N-1} \quad \text{if } N > 0 \tag{1.25}$$

$$C_N := f_{-1} \cdots f_N \quad \text{if } N < 0 \tag{1.26}$$

$$C_N := 1 \quad \text{if } N = 0 \tag{1.27}$$

and

$$|N\rangle := \bar{C}_N|0\rangle \tag{1.28}$$

where

$$\bar{C}_N := f_{N-1} \cdots f_0 \quad \text{if } N > 0 \tag{1.29}$$

$$\bar{C}_N := \bar{f}_N \cdots \bar{f}_{-1} \quad \text{if } N < 0 \tag{1.30}$$

$$\bar{C}_N := 1 \quad \text{if } N = 0 \tag{1.31}$$

The states (1.24) and (1.28) are referred to as the left and right charged vacuum vectors, respectively, with charge N.

In what follows we use the notational convention

$$\Delta_N(x) = \det\,(x_i^{N-k})|_{i,k=1,\dots,N}\ (N>0), \quad \Delta_0(x) = 1, \quad \Delta_N(x) = 0\ (N<0). \tag{1.32}$$

From the relations

$$\langle 0|\bar{f}_{N-k} f(x_i)|0\rangle = x_i^{N-k}, \quad \langle 0|f_{-N+k-1}\bar{f}(y_i)|0\rangle = y_i^{N-k}, \quad k = 1, 2, \dots, \tag{1.33}$$

and (1.23), it follows that

$$\langle N|f(x_1)\cdots f(x_n)|0\rangle = \delta_{n,N}\Delta_N(x), \quad N \in \mathbf{Z}, \tag{1.34}$$

$$\langle -N|\bar{f}(y_1)\cdots \bar{f}(y_n)|0\rangle = \delta_{n,N}\Delta_N(y), \quad N \in \mathbf{Z}. \tag{1.35}$$

Following [9], [10] we consider $\hat{GL}_\infty$ element

$$g = e^h, \quad h = \sum_{i,j} h_{i,j} f_i \bar{f}_j, \quad h_{i,j} \in \mathbb{C} \tag{1.36}$$

Via the conjugation, $(\cdot) \to g(\cdot)g^{-1}$, each $g \in \hat{GL}_\infty$ acts on the spaces $(\oplus_{m\in\mathbb{Z}}\mathbb{C}f_m)$ and $\left(\oplus_{m\in\mathbb{Z}}\mathbb{C}\bar{f}_m\right)$ as linear transformations [9], [10].

We suppose that the following factorization condition is valid:

$$g = g_+ g_-, \quad \langle 0|g^+ = \langle 0|, \quad g_-|0\rangle = |0\rangle, \tag{1.37}$$

where $g_+, g_- \in \hat{GL}_\infty$.

Remark. Though, the property (1.37) is valid for a rather wide class of (1.36) (which includes all cases when the sum in (1.36) is finite) , however, we do not know the general theorem providing sufficient and necessary conditions to have this property in case the sum in (1.36) is infinite.

Consider

$$\langle 0|v_N \cdots v_1 g \bar{v}_1 \cdots \bar{v}_N|0\rangle,$$

where each $v_i \in (\oplus_{m\in\mathbb{Z}}\mathbb{C}f_m)$ and each $\bar{v}_i \in \left(\oplus_{m\in\mathbb{Z}}\mathbb{C}\bar{f}_m\right)$, $i = 1,\ldots,N$. Denoting $w_i = (g_+)^{-1}v_i g_+ \in (\oplus_{m\in\mathbb{Z}}\mathbb{C}f_m)$ and $\bar{w}_i = (g_-)\bar{v}_i(g_-)^{-1} \in \left(\oplus_{m\in\mathbb{Z}}\mathbb{C}\bar{f}_m\right)$ we have

$$\langle 0|v_N\cdots v_1 g\bar{v}_1\cdots\bar{v}_N|0\rangle = \langle 0|w_N\cdots w_1\bar{w}_1\cdots\bar{w}_N|0\rangle = \det\langle 0|w_i\bar{w}_j|0\rangle|_{i,j=1,\ldots,N}$$

where the second equality is due to the Wick theorem (1.23). Thus

$$\langle 0|v_N\cdots v_1 g\bar{v}_1\cdots\bar{v}_N|0\rangle = \det\langle 0|v_i g\bar{v}_j\cdots|0\rangle|_{i,j=1,\ldots,N} \tag{1.38}$$

1.2 Multi-component fermions

One obtains the so-called p-component fermion formalism by re-numerating the above free fermions (1.13) as follows

$$f_n^{(\alpha)} := f_{pn+\alpha-1}\ , \quad \bar{f}_n^{(\alpha)} := \bar{f}_{pn+\alpha-1}\ , \tag{1.39}$$

$$f^{(\alpha)}(z) := \sum_{k=-\infty}^{+\infty} z^k f_k^{(\alpha)}\ , \quad \bar{f}^{(\alpha)}(z) := \sum_{k=-\infty}^{+\infty} z^{-k-1}\bar{f}_k^{(\alpha)}\ , \tag{1.40}$$

where $\alpha = 1,\ldots,p$. From (1.13) we obviously have

$$[f_n^{(\alpha)}, f_m^{(\beta)}]_+ = [\bar{f}_n^{(\alpha)}, \bar{f}_m^{(\beta)}]_+ = 0, \quad [f_n^{(\alpha)}, \bar{f}_m^{(\beta)}]_+ = \delta_{\alpha,\beta}\delta_{n,m}. \tag{1.41}$$

Right and left vacuum vectors are respectively defined

$$|\underbrace{0,\ldots,0}_{p}\rangle := |0\rangle, \quad \langle\underbrace{0,\ldots,0}_{p}| := \langle 0| \tag{1.42}$$

where $|0\rangle$ and $\langle 0|$ were introduced in (1.19).

As it follows from (1.19)

$$f_m^{(\alpha)}|0,\ldots,0\rangle = 0 \qquad (m<0), \qquad \bar{f}_m^{(\alpha)}|0,\ldots,0\rangle = 0 \qquad (m\geq 0), \tag{1.43}$$

$$\langle 0,\ldots,0|f_m^{(\alpha)} = 0 \qquad (m\geq 0), \qquad \langle 0,\ldots,0|\bar{f}_m^{(\alpha)} = 0 \qquad (m<0). \tag{1.44}$$

We also introduce the states

$$\langle n^{(1)},\ldots,n^{(p)}| := \langle 0,0|C_{n^{(1)}}\cdots C_{n^{(p)}} \tag{1.45}$$

where

$$C_{n^{(\alpha)}} := \bar{f}_0^{(\alpha)}\cdots\bar{f}_{n^{(\alpha)}-1}^{(\alpha)} \quad \text{if } n^{(\alpha)}>0 \tag{1.46}$$

$$C_{n^{(\alpha)}} := f_{-1}^{(\alpha)}\cdots f_{n^{(\alpha)}}^{(\alpha)} \quad \text{if } n^{(\alpha)}<0 \tag{1.47}$$

$$C_{n^{(\alpha)}} := 1 \quad \text{if } n^{(\alpha)}=0 \tag{1.48}$$

$$|n^{(1)},\dots,n^{(p)}\rangle := \bar{C}_{n^{(p)}}\cdots\bar{C}_{n^{(1)}}|0,0\rangle \tag{1.49}$$

where

$$\bar{C}_{n^{(\alpha)}} := f^{(\alpha)}_{n^{(\alpha)}-1}\cdots f^{(\alpha)}_0 \quad \text{if } n^{(\alpha)}>0 \tag{1.50}$$

$$\bar{C}_{n^{(\alpha)}} := \bar{f}^{(\alpha)}_{n^{(\alpha)}}\cdots \bar{f}^{(\alpha)}_{-1} \quad \text{if } n^{(\alpha)}<0 \tag{1.51}$$

$$\bar{C}_{n^{(\alpha)}} := 1 \quad \text{if } n^{(\alpha)}=0 \tag{1.52}$$

Let us call (1.45) and (1.49) respectively left and right charged vacuum vectors with the charge $(n^{(1)},\dots,n^{(p)})$.

We easily verify that

$$f^{(\alpha)}_m|*,n^{(\alpha)},*\rangle=0 \qquad (m<n^{(\alpha)}), \qquad \bar{f}^{(1)}_m|*,n^{(\alpha)},*\rangle=0 \qquad (m\geq n^{(\alpha)}), \tag{1.53}$$

$$\langle *,n^{(\alpha)},*|f^{(\alpha)}_m=0 \qquad (m\geq n^{(\alpha)}), \qquad \langle *,n^{(\alpha)},*|\bar{f}^{(\alpha)}_m=0 \qquad (m< n^{(\alpha)}), \tag{1.54}$$

where $*$ serve for irrelevant components in vacuum vectors.

Remark 1.1. *For calculations we use the Wick theorem in form (1.23). There are two ways to do it:*

(1) The first one is to use (1.23) just remembering that p-component fermions are composed of usual ones, see (1.39).

(2) The second way is to use formula (1.23) separately for each component. Namely, to calculate the vacuum expectation value of an operator O, first, we present it in form

$$O=\sum_i O_i^{(1)}\cdots O_i^{(p)} \tag{1.55}$$

Then

$$\langle 0|O|0\rangle = \sum_i \langle 0|O_i^{(1)}\cdots O_i^{(p)}|0\rangle = \sum_i \langle 0|O_i^{(1)}|0\rangle\cdots\langle 0|O_i^{(p)}|0\rangle \tag{1.56}$$

where the Wick theorem in form (1.23) is applied to each of $\langle 0,\dots,0|O_i^{(\alpha)}|0,\dots,0\rangle$.

2 Fermionic representation for Z_N

Consider the element of the Clifford algebra of the following form

$$g=e^{A_1}g_2e^{A_2}g_3\cdots e^{A_{p-2}}g_{p-1}e^{A_{p-1}}, \tag{2.1}$$

where

$$A_\alpha = \int\int f^{(\alpha)}(x)\bar{f}^{(\alpha+1)}(y)d\mu_\alpha(x,y), \quad \alpha = 1,\dots,p-1, \tag{2.2}$$

with measure $d\mu_\alpha(x,y)$, which we do not specify.

In (2.1)

$$g_\alpha = e^{h_\alpha}, \quad h_\alpha = \sum_{i,j} h_{i,j}^{(\alpha)} f_i^{(\alpha)} \bar{f}_j^{(\alpha)}, \quad h_{i,j}^{(\alpha)} \in \mathbb{C}, \tag{2.3}$$

so that we have

$$f_i^{(\beta)} g_\alpha = g_\alpha f_i^{(\beta)}, \quad \alpha \neq \beta, \quad i \in \mathbb{Z} \tag{2.4}$$

We also suppose that each $g_\alpha = e^{h^{(\alpha)}}$, $\alpha = 2,\dots,p-1$, may be factorized into $\hat{GL}_\infty^{(\alpha)}$ elements g_α^+ and g_α^- as follows (see (1.37))

$$g_\alpha = g_\alpha^+ g_\alpha^-, \quad \langle *, \overset{\alpha}{\hat{0}}, * | g_\alpha^+ = \langle *, \overset{\alpha}{\hat{0}}, * |, \quad g_\alpha^- | *, \overset{\alpha}{\hat{0}}, * \rangle = | *, \overset{\alpha}{\hat{0}}, * \rangle \tag{2.5}$$

where by $*$ we denote irrelevant components of a vacuum vector (different from the component α marked by hats).

Now, let us notice that by (1.38) we have

$$\langle 0|\bar{f}^{(\alpha)}(y_1)\cdots\bar{f}^{(\alpha)}(y_N)g_\alpha f^{(\alpha)}(x_N)\cdots f^{(\alpha)}(x_1)|0\rangle = \det\left(\langle 0|\bar{f}^{(\alpha)}(y_i)g_\alpha f^{(\alpha)}(x_j)|0\rangle\right)_{i,j=1,\dots,N} \tag{2.6}$$

Now let us prove, that for special choice of functions ρ_α, $\alpha = 2,\dots,p-1$, namely, for

$$\langle 0|\bar{f}^{(\alpha)}(y)g_\alpha f^{(\alpha)}(x)|0\rangle = \rho_\alpha(y,x) \tag{2.7}$$

we have

$$(N!)^{p-1}\langle N,0,\dots,0,-N|g|0,0,\dots,0,0\rangle = Z_N \tag{2.8}$$

Indeed, to get a non-vanishing expectation value in the left hand side, we have to pick up only N-th term, $\frac{A_1^N}{N!}$, in the Taylor series for e^{A_1} (this is because e^{A_1} is the only factor of g which contains the first component fermions, and the matrix element $\langle N,*|A_1^n|0,*\rangle \equiv 0$ until $n = N$). Using the known formula (1.34), we obtain, that the left hand side of (2.8) is equal to the integral

$$\int d\mu_1(x_1^{(1)},y_1^{(2)})\dots\int d\mu_1(x_N^{(1)},y_N^{(2)})\Delta_N(x^{(1)})R_1,$$

$$R_1 = \langle *,0,\dots,0,-N|\bar{f}^{(2)}(y_1^{(2)})\cdots\bar{f}^{(2)}(y_N^{(2)})g_2\cdots|*,0,\dots,0\rangle$$

where we put $*$ on the first place of the left and right vacuum vectors to show that we forget about the first component fermions. This is the first step.

Then, we have to pick up only N-th term, $\frac{A_2^N}{N!}$, when developing the next factor e^{A_2}. Otherwise, the vacuum expectation values of the second component fermions vanishes. This is because the second component fermions are in presence only in e^{A_1}, g_2 and in e^{A_2} factors of g, and the g_2 is a sum of monomials, each of which contains equal number of $f^{(2)}$ and $\bar{f}^{(2)}$ fermions, while e^{A_1} contains only $f^{(2)}$, and e^{A_2} contains only $\bar{f}^{(2)}$ fermions. Thus, second component fermions yields the expression

$$\bar{f}^{(2)}(y_1^{(2)})\cdots\bar{f}^{(2)}(y_N^{(2)})g_2 f^{(2)}(x_1^{(2)})\cdots f^{(2)}(x_N^{(2)})$$

which should be integrated with measures $\prod_{i=1}^N d\mu(*,y_i^{(2)})d\mu(x_i^{(2)},*)$, and then substituted inside $\langle N,0,\dots,0,-N|$ and $|0,0,\dots,0,0\rangle$. Denoting

$$\langle 0|\bar{f}^{(2)}(y_1^{(2)})\cdots\bar{f}^{(2)}(y_N^{(2)})g_2 f^{(2)}(x_1^{(2)})\cdots f^{(2)}(x_N^{(2)})|0\rangle = \varrho_2(y^{(2)},x^{(2)}) \tag{2.9}$$

(which, by (2.6), is equal to $\det\rho_2(y_i^{(2)},x_j^{(2)})$) we obtain that the l.h.s of (2.8) is equal to the integral

$$\int d\mu_1(x_1^{(1)},y_1^{(2)})\dots\int d\mu_1(x_N^{(1)},y_N^{(2)})\Delta_N(x^{(1)})$$
$$\int d\mu_2(x_1^{(2)},y_1^{(3)})\dots\int d\mu_2(x_N^{(2)},y_N^{(3)})\varrho_3(y^{(3)},x^{(3)})R_2,$$
$$R_2=\langle *,*,0,\dots,0,-N|\bar{f}^{(2)}(y_1^{(2)})\cdots\bar{f}^{(2)}(y_N^{(2)})g_2\cdots|*,*,0,\dots,0\rangle$$

where we put $*$ on the first and second places of the left and right vacuum vectors to show that we forget about the first and the second component fermions. This is the second step.

Then, it is easy to see that each exponential e^{A_α} should be replaced by their N-th Taylor term, otherwise the l.h.s. of (2.8) vanishes, it means we have

$$\langle N,0,\dots,0,-N|g|0,0,\dots,0,0\rangle =$$
$$\frac{1}{(N!)^{p-1}}\langle N,0,\dots,0,-N|A_1^N g_2 A_2^N\cdots g_{p-1}A_{p-1}^N|0,0,\dots,0,0\rangle \tag{2.10}$$

Continuing excluding step by step third- forth- and so on component fermions, and, on the last step, using the known formula (1.35), we obtain that (2.10) is equal to (1.3).

At last we want to make the following remark

Remark 2.1. Insert additional factors to (2.1) as follows

$$g = e^{A_1} g_2 e^{A_2} g_3 \cdots e^{A_{p-2}} g_{p-1} e^{A_{p-1}} \to g_{\circlearrowleft} := e^{A_1} g_2 e^{A_2} g_3 \cdots e^{A_{p-2}} g_{p-1} e^{A_{p-1}} g_p g_1 e^{A_p} \tag{2.11}$$

where

$$g_\alpha = e^{\sum_{i,j} h_{i,j}^{(\alpha)} f_i^{(\alpha)} \bar{f}_j^{(\alpha)}}, \quad A_\alpha = \int\int f^{(\alpha)}(x) \bar{f}^{(\alpha+1)}(y) d\mu_\alpha(x,y), \quad \alpha = 1, \dots, p \tag{2.12}$$

where $\bar{f}^{(p+1)}(y) \equiv \bar{f}^{(1)}(y)$. (Thus we add g_1, g_p and $d\mu_p(x,y)$ to our collection of data, g_α, $\alpha = 2, \dots, p-1$ and $d\mu_\alpha(x,y)$, $\alpha = 1, \dots, p-1$). Then

$$\langle 0, 0, \dots, 0, 0 | g_{\circlearrowleft} | 0, 0, \dots, 0, 0 \rangle = \sum_{N=0}^{\infty} c_N Z_N^{\circlearrowleft} \tag{2.13}$$

where c_N are certain numbers and each $Z_N^{\circlearrowleft}$ is the following integral over $2pN$ variables $x^{(\alpha)} = (x_1^{(\alpha)}, \dots, x_N^{(\alpha)})$ and $y^{(\alpha)} = (y_1^{(\alpha)}, \dots, y_N^{(\alpha)})$, $\alpha = 1, \dots, p$:

$$Z_N^{\circlearrowleft} = \int \prod_{\alpha=1}^{p} \varrho_\alpha(y^{(\alpha)}, x^{(\alpha)}) \prod_{\alpha=1}^{p} d\mu_\alpha(x^{(\alpha)}, y^{(\alpha+1)}), \quad y^{(p+1)} \equiv y^{(1)} \tag{2.14}$$

(notice that variables $y^{(1)}$ and $x^{(p)}$ are not fixed by (1.1))

In (2.14) $d\mu_\alpha(x^{(\alpha)}, y^{(\alpha+1)})$ $(\alpha = 1, \dots, p-1)$ are defined by (1.2) and

$$d\mu_p(x^{(p)}, y^{(1)}) := \prod_{i=1}^{N} d\mu_p(x_i^{(p)}, y_i^{(1)}), \tag{2.15}$$

and each $\varrho_\alpha(y^{(\alpha)}, x^{(\alpha)})$ is defined by (2.6)-(2.7), where now $\alpha = 1, \dots, p$.

Sums (2.13) and their relation to the grand partition function of *closed* chains of coupled random matrices and to integrable equations will be considered in a forthcoming paper.

3 Deformation of measure and relations to integrable hierarchies

The described deformation

$$d\mu_1(x,y) \to d\mu_1(x,y|\mathbf{t}^{(1)}, n, \bar{\mathbf{t}}^{(1)}) := x^{n_1} e^{V(x,\mathbf{t}^{(\alpha)}) + V(x^{-1}, \bar{\mathbf{t}}^{(1)})} d\mu_1(x,y), \tag{3.1}$$

$$d\mu_{p-1}(x,y) \to d\mu_{p-1}(x,y|\mathbf{t}^{(p)}, n, \bar{\mathbf{t}}^{(p)}) := y^{n_p} e^{V(y,\mathbf{t}^{(p)}) + V(y^{-1}, \bar{\mathbf{t}}^{(p)})} d\mu_{p-1}(x,y) \tag{3.2}$$

$$V(x, \mathbf{t}^{(\alpha)}) = \sum_{m=1}^{\infty} x^m t_m^{(\alpha)}, \quad V(x^{-1}, \bar{\mathbf{t}}^{(\alpha)}) = \sum_{m=1}^{\infty} x^{-m} \bar{t}_m^{(\alpha)}, \quad \alpha = 1, p, \tag{3.3}$$

Then, it is quite known fact that in this case $Z_N = \tau_N(\mathbf{t}^{(1)}, \bar{\mathbf{t}}^{(p)})$, where $\tau_N(\mathbf{t}^{(1)}, \bar{\mathbf{t}}^{(p)})$ is a tau function of the (one-component) TL hierarchy. Indeed, one just re-writes (1.3) as $2N$-fold integral with a modified measure $d\mu^{mod}$ (the latter depends on the choice of ρ_α) :

$$\int \prod_{i=1}^{N} d\mu^{mod}(x_i^{(1)}, y_i^{(p)}) x_i^{n_1} y_i^{n_p} e^{V(x_i^{-1}, \bar{\mathbf{t}}^{(1)}) + V(y_i^{-1}, \bar{\mathbf{t}}^{(p)})} e^{V(x_i^{(1)}, \mathbf{t}^{(1)}) + V(y_i^{(p)}, \mathbf{t}^{(p)})} \Delta_N(x^{(1)}) \Delta_N(y^{(p)})$$

Moreover, as a function of $\mathbf{t}^{(1)}, \mathbf{t}^{(p)}, \bar{\mathbf{t}}^{(1)}, \bar{\mathbf{t}}^{(p)}$, the integral $Z_N(\mathbf{t}^{(1)}, \mathbf{t}^{(p)}, n_1, n_p, \bar{\mathbf{t}}^{(1)}, \bar{\mathbf{t}}^{(p)})$ is a tau function of the coupled two-component KP, or, the same, a tau function of the two-component TL hierarchy, see [1].

Now, in addition, we consider the following deformations of functions ρ_α, $\alpha = 2, \dots, p-1$,

$$\rho_\alpha(x, y) \to \rho_\alpha(x, y | \mathbf{t}^{(\alpha)}, n^{(\alpha)}, \bar{\mathbf{t}}^{(\alpha)}) := \tag{3.4}$$

$$\langle n^{(\alpha)} | e^{H^{(\alpha)}(\mathbf{t}^{(\alpha)})} \bar{f}^{(\alpha)}(y_1^{(\alpha)}) \cdots \bar{f}^{(\alpha)}(y_N^{(\alpha)}) g_\alpha f^{(\alpha)}(x_1^{(\alpha)}) \cdots f^{(\alpha)}(x_N^{(\alpha)}) e^{\bar{H}^{(\alpha)}(\bar{\mathbf{t}}^{(\alpha)})} | n^{(\alpha)} \rangle \tag{3.5}$$

where $\mathbf{t}^{(\alpha)} = (t_1^{(\alpha)}, t_1^{(\alpha)}, \dots)$ and $\bar{\mathbf{t}}^{(\alpha)} = (\bar{t}_1^{(\alpha)}, \bar{t}_2^{(\alpha)}, \dots)$ are the deformation parameters, and where the "Hamiltonians" $H_k^{(\alpha)}$, $k = \pm 1, \pm 2, \dots$, are defined by

$$H^{(\alpha)}(\mathbf{t}^{(\alpha)}) = \sum_{k=1}^{\infty} H_k^{(\alpha)} t_k^{(\alpha)}, \quad \bar{H}^{(\alpha)}(\bar{\mathbf{t}}^{(\alpha)}) = \sum_{k=1}^{\infty} H_{-k}^{(\alpha)} \bar{t}_k^{(\alpha)}, \quad H_k^{(\alpha)} = \sum_{n=-\infty}^{+\infty} f_n^{(\alpha)} \bar{f}_{n+k}^{(\alpha)} \tag{3.6}$$

(for future purpose we define them for $\alpha = 1, \dots, p$ range).

Let us note that the expectation value (3.5), by definition [9], [10], [15], is a tau function of one component TL and in our case may be denoted by

$$\tau_{n^{(\alpha)}}(\mathbf{t}^{(\alpha)} + [x^{(\alpha)}], \bar{\mathbf{t}}^{(\alpha)} + [y^{(\alpha)}])$$

where

$$[x] := \left(\frac{x}{1}, \frac{x^2}{2}, \frac{x^3}{3}, \dots \right)$$

Now let us prove that the combination of deformations (3.1)-(3.2) and (3.4) is equivalent to the replacement

$$\langle N, 0, \dots, 0, -N | g | 0, 0, \dots, 0, 0 \rangle \to$$

$$\tau_N(\mathbf{t}, \mathbf{n}, \bar{\mathbf{t}}) := \langle N + n^{(1)}, n^{(2)}, \dots, n^{(p-1)}, -N - n^{(p)} | e^{H(\mathbf{t})} g e^{\bar{H}(\bar{\mathbf{t}})} | n^{(1)}, n^{(2)}, \dots, n^{(p-1)}, -n^{(p)} \rangle, \tag{3.7}$$

where

$$H(\mathbf{t}) = \sum_{\alpha=1}^{p} \sum_{k=1}^{\infty} H_k^{(\alpha)} t_k^{(\alpha)}, \quad \bar{H}(\bar{\mathbf{t}}) = \sum_{\alpha=1}^{p} \sum_{k=1}^{\infty} H_{-k}^{(\alpha)} \bar{t}_k^{(\alpha)}$$

the "Hamiltonians" $H_k^{(\alpha)}$ were defined earlier by (3.6).

Proof. Indeed, we have that each terms of type (2.9), namely, each

$$\langle 0|\bar{f}^{(\alpha)}(y_1^{(\alpha)}) \cdots \bar{f}^{(\alpha)}(y_N^{(\alpha)}) g_\alpha f^{(\alpha)}(x_1^{(\alpha)}) \cdots f^{(\alpha)}(x_N^{(\alpha)})|0\rangle = \varrho(y^{(\alpha)}, x^{(\alpha)}), \quad \alpha = 2, \ldots, p-1, \tag{3.8}$$

is now replaced by

$$\langle n^{(\alpha)}|e^{H^{(\alpha)}(\mathbf{t}^{(\alpha)})} \bar{f}^{(\alpha)}(y_1^{(\alpha)}) \cdots \bar{f}^{(\alpha)}(y_N^{(\alpha)}) g_\alpha f^{(\alpha)}(x_1^{(\alpha)}) \cdots f^{(\alpha)}(x_N^{(\alpha)}) e^{\bar{H}^{(\alpha)}(\bar{\mathbf{t}}^{(\alpha)})}|n^{(\alpha)}\rangle \tag{3.9}$$

which is, by definition [9], [10], [15], a tau function of one component TL. Due to (1.38) it is equal to

$$\det \rho_\alpha(x_i, y_j|\mathbf{t}, \mathbf{n}, \bar{\mathbf{t}})$$

where

$$\rho_\alpha(x_i, y_j|\mathbf{t}, \mathbf{n}, \bar{\mathbf{t}}) := \langle n^{(\alpha)}|e^{H^{(\alpha)}(\mathbf{t}^{(\alpha)})} \bar{f}^{(\alpha)}(y_i^{(\alpha)}) g_\alpha f^{(\alpha)}(x_j^{(\alpha)}) e^{\bar{H}^{(\alpha)}(\bar{\mathbf{t}}^{(\alpha)})}|n^{(\alpha)}\rangle$$

As for $\alpha = 1, p$ we have

$$\langle N, *|e^{H^{(1)}(\mathbf{t}^{(1)})} f(x_1) \cdots f(x_N) e^{\bar{H}^{(1)}(\bar{\mathbf{t}}^{(1)})}|0, *\rangle = a_1 \Delta_N(x) e^{\sum_{i=1}^N V(x_i, \mathbf{t}^{(1)}) + V(x_i^{-1}, \bar{\mathbf{t}}^{(1)})}, \tag{3.10}$$

$$\langle *, 0|e^{\bar{H}^{(p)}(\bar{\mathbf{t}}^{(p)})} \bar{f}(y_N) \cdots \bar{f}(y_1) e^{\bar{H}^{(p)}(\bar{\mathbf{t}}^{(p)})}|*, -N\rangle = a_p \Delta_N(y) e^{\sum_{i=1}^N V(y_i, \mathbf{t}^{(p)}) + V(y_i^{-1}, \bar{\mathbf{t}}^{(p)})}, \tag{3.11}$$

where $a_\alpha = e^{\sum_{k=1}^{\infty} k t_k^{(\alpha)} \bar{t}_k^{(\alpha)}}$, which contribute to the deformation respectively of $d\mu_1$ and of $d\mu_{p-1}$. The end of proof.

Thus, we obtain that the deformation of functions $\rho_\alpha, \alpha = 2, \ldots, p-1$, and also of $d\mu_1, d\mu_{p-1}$ reduce to the fact that Z_N is equal to $\tau_N(\mathbf{t}, \mathbf{n}, \bar{\mathbf{t}})$. It is known [9], [10], [14] that thus constructed $\tau_N(\mathbf{t}, \mathbf{n}, \bar{\mathbf{t}})$ is a tau function of the coupled p-component KP hierarchy, or, the same, p-component TL hierarchy.

4 Conclusion

We equate the multi-integral (1.3) to the fermionic expectation value (2.8). On the one hand we hope that the fermionic representation allows to evaluate different magnitudes related to the matrix models, like spectral determinants (compare with [2]), or perturbative series generalizing [12], [13]. On the other hand it allows to incorporate the study of these integrals and related multi-matrix models to the study of multi-component integrable hierarchies

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